

Computing Injective Resolutions and Local Cohomology over Affine Semigroup Rings*

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Abstract

We present an OSCAR implementation of the Helm–Miller algorithms for computing Q -graded injective and irreducible resolutions, irreducible decompositions, and local cohomology of modules over an affine semigroup ring $k[Q]$. The $\mathbb{Z}^d$ -grading of $k[Q]$ yields a combinatorial encoding of the otherwise infinitely generated injective modules, enabling effective computation. We also correct an error in one of the original algorithms.

1 Introduction

Free resolutions over polynomial rings are computationally well-developed. *Injective* resolutions are harder: indecomposable injectives are typically infinitely generated. Over an affine semigroup ring $k[Q]$ the $\mathbb{Z}^d$ -grading enables a combinatorial encoding that makes injective resolutions effectively computable. Helm and Miller [2] gave algorithms for Q -graded injective and irreducible resolutions and for local cohomology. Beyond classical questions such as the Cohen–Macaulay property, these structures have recently appeared in multiparameter persistence and topological data analysis [3, 1].

We present an OSCAR implementation of these algorithms, available as the experimental module `InjectiveResolutions`. It computes minimal irreducible and injective resolutions, irreducible decompositions, and sector partitions of local cohomology. We also correct an error in [2, Alg. 3.6], which misses socle elements supported on multiple generators. Currently only normal semigroups are supported in OSCAR. Our fork <https://github.com/annahofer00/Oscar.jl.git> extends the computation to non-normal semigroups, has further improvements, and will be merged soon.

2 Preliminaries

We follow the notation and conventions of [4, Ch. 11] throughout. An *affine semigroup* is a finitely generated submonoid $Q \subseteq \mathbb{Z}^d$. Given a field k , the *monoid algebra* $k[Q]$ is the k -algebra with k -basis $\{x^a : a \in Q\}$ and multiplication $x^a x^b = x^{a+b}$. More generally, for a subset $S \subseteq \mathbb{Z}^d$ we write $k\{S\}$ for the $\mathbb{Z}^d$ -graded k -vector space with basis $\{x^a : a \in S\}$ and multiplication $x^a x^b = x^{a+b}$ if $a + b \in S$,

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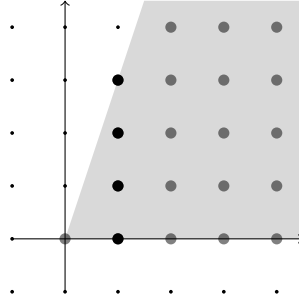


Figure 1: polyhedral cone $\mathbb{R}_{\geq 0}Q$ (gray region); generators of Q (black); other points in Q (gray)

and zero otherwise. The semigroup Q is *pointed* if $Q \cap (-Q) = \{0\}$ and *normal* if $Q = \mathbb{Z}Q \cap \mathbb{R}_{\geq 0}Q$. A *face* of Q is a subset $F \subseteq Q$ with $a + b \in F$ if and only if $a, b \in F$. Faces correspond bijectively to $\mathbb{Z}^d$ -graded prime ideals $P_F = k\{Q \setminus F\}$ of $k[Q]$. Up to isomorphism, the *indecomposable injectives* in the category of $\mathbb{Z}^d$ -graded $k[Q]$ -modules are of the form $k\{a + F - Q\}$ for a face F and $a \in \mathbb{Z}^d$, and every $\mathbb{Z}^d$ -graded injective is a direct sum of these. A $k[Q]$ -module is *Q -graded* if it vanishes outside Q , and an *injective resolution* of a Q -graded M is an exact sequence $0 \rightarrow M \rightarrow J^0 \rightarrow J^1 \rightarrow \dots$, where each J^i is injective.

Indecomposable injectives are typically not finitely generated, but their Q -graded parts are quotients $k[Q]/W$, where W is an *irreducible ideal*, i.e., W is not expressible as an intersection of two strictly larger ideals. An *irreducible resolution* is a graded exact sequence $0 \rightarrow M \rightarrow \overline{W}^0 \rightarrow \overline{W}^1 \rightarrow \dots$ with $\overline{W}^i = \bigoplus_j k[Q]/W_{ij}$ and each W_{ij} irreducible. Every finitely generated Q -graded M has a finite minimal irreducible resolution, unique up to isomorphism, which is the Q -graded part of a minimal injective resolution of M . We exploit this finite, finitely generated counterpart to compute injective resolutions. For an ideal $I \subseteq k[Q]$ the *i -th local cohomology module of M supported on I* is $H_I^i(M) = H^i(\Gamma_I(E^\bullet))$, where $E^\bullet$ is an injective resolution of M and $\Gamma_I(N) = \{n \in N : n \cdot I^t = 0 \text{ for some } t\}$. In the $\mathbb{Z}^d$ -graded setting, Γ_I retains exactly the indecomposable summands $k\{a + F - Q\}$ with $I \subseteq P_F$.

3 An example

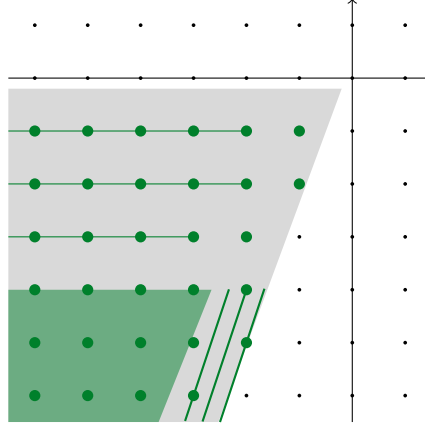
Consider the monoid $Q = \mathbb{N}\{(1,0), (1,1), (1,2), (1,3)\} \subset \mathbb{Z}^2$. Its monoid algebra

$$k[Q] \cong k[x_1, x_2, x_3, x_4] / (x_3^2 - x_2x_4, x_2x_3 - x_1x_4, x_2^2 - x_1x_3)$$

is normal, since all lattice points in the cone $\mathbb{R}_{\geq 0}Q$ belong to Q (see Figure 1). The generators of the binomial prime ideal presenting the algebra are known as a *Markov basis* in algebraic statistics. We define this in OSCAR and check that $k[Q]$ is normal:

```
julia> kQ = monoid_algebra([[1,0],[1,1],[1,2],[1,3]], QQ); is_normal(kQ)
true
```

The implementation runs over any field for which OSCAR provides Gröbner-basis and free-resolution computations. We compute the first steps in an injective resolution of $M = k[Q]$, the monoid algebra as a Q -graded module over itself.


 Figure 2: non-zero local cohomology sectors of $H_{\mathfrak{m}}^2(k[Q])$

```

julia> KQ = quotient_ring_as_module(ideal(kQ, []));
julia> inj_res = injective_resolution(KQ, 1)
injective resolution
  J^0 -> J^1
where
  J^0 = direct sum of
    k[[0, 0] + F - Q], where p_F = Ideal ()
  J^1 = direct sum of
    k[[0, -1] + F - Q], where p_F = Ideal (x_2, x_3, x_4)
    k[[-1, -2] + F - Q], where p_F = Ideal (x_1, x_2, x_3)
    
```

The output reads as follows. The injective hull J^0 is the single indecomposable $k\{Q - Q\}$, corresponding to the face $F = Q$ (prime $P_F = 0$) with shift $a = (0, 0)$, since $k[Q]$ is torsion-free over itself. Then J^1 is a sum of two indecomposable injectives whose associated primes (x_2, x_3, x_4) and (x_1, x_2, x_3) are the two height-one primes of $k[Q]$, one for each ray of the cone $\mathbb{R}_{\geq 0}Q$. The shift vectors $a = (0, -1)$ and $a = (-1, -2)$ record the $\mathbb{Z}^2$ -graded degrees at which the corresponding Bass numbers are non-zero. Generally, these shifts are only well-defined modulo $\mathbb{Z}F$. For the latter summand, we have $\mathbb{Z}F = \mathbb{Z} \cdot (1, 3)$, and a more natural representative of the same coset would be $a = (0, 1)$, the unique lift with smallest entries. The value $(-1, -2)$ printed by OSCAR reflects an internal choice of representative arising from the way certain socle elements are computed.

Moving on to local cohomology, a *sector partition* of $H_I^i(M)$ decomposes $\mathbb{Z}^d$ into finitely many *sectors* on which $H_I^i(M)$ is constant, giving a finite encoding of the otherwise infinitely presented module [2, Definition 1.2].

We compute a sector partition of the local cohomology module $H_{\mathfrak{m}}^2(k[Q])$, where $\mathfrak{m} = (x_1, x_2, x_3, x_4)$ is the graded maximal ideal.

```

julia> m = ideal(kQ, gens(kQ)); H2 = local_cohomology(KQ, m, 2);
julia> length(filter(!is_zero, H2.sectors)) # 10
    
```

The 10 non-zero sectors can now be examined, for instance, the first is a single lattice point:

```

julia> nz = filter(!is_zero, H2.sectors); nz[1]
sector S_A of ZZ^2 of dimension 0 with
  A = [[], [2], []] and local cohomology of dimension 1
    
```

```
julia> lattice_points(nz[1].sector)
1-element SubObjectIterator{PointVector{ZZRingElem}}:
 [-1, -2]
```

The full sector partition appears as the green polyhedral regions in Figure 2. Each sector is supported on lattice points in $-\text{relint}(\mathbb{R}_{\geq 0}Q)$, depicted in gray. The graded pieces on these sectors are one-dimensional, i.e., $H_{\mathfrak{m}}^2(k[Q])_d \cong k$ for all $d \in -\text{relint}(\mathbb{R}_{\geq 0}Q) \cap \mathbb{Z}Q$.

4 Implementation

Data structures. Our `MonoidAlgebra` wraps a graded polynomial or quotient ring together with the combinatorics of Q : cone, faces, and $\mathbb{Z}^d$ -graded primes. An indecomposable injective $k\{a + F - Q\}$ is stored as the pair (F, a) , an injective module as a list of such pairs, and a map between them as a *monomial matrix* indexed by face/degree pairs [4, Def. 11.18].

Main functions. `injective_resolution(M, i)` (min. injective resolution up to cohomological degree i), `irreducible_resolution(M)`, `local_cohomology(M, I, i)` (sector partition of $H_I^i(M)$, requires $k[Q]$ normal), and `irreducible_decomposition(I)` for a $\mathbb{Z}^d$ -graded ideal $I \subseteq k[Q]$.

Correction to Algorithm 3.6 of [2]. The original algorithm processes one generator g of M at a time, reading off the coefficient of a socle element (an element annihilated by P_F) from $(0 :_{(g)} P_F)$. This misses socle elements where no single generator is annihilated by P_F . Our correction processes the socle in each $\mathbb{Z}F$ -coset $\bar{a}$ jointly. The valid coefficient vectors form the kernel $K = \ker(C)$, where C has as rows the coefficient vectors of the *relevant* relations of M (those r with $(\deg r + Q) \cap (\bar{a} + \mathbb{Z}F) \neq \emptyset$). For a k -basis $b_1, \dots, b_s$ of the socle $(0 :_M P_F)$ in degree $\bar{a}$, we pick $\lambda_1, \dots, \lambda_s \in K$ dual to the coefficient vectors of the b_i , giving an injective map of the socle into the irreducible hull.

Performance. The main bottleneck of our implementation is the repeated Gröbner basis computation needed in various places, for example when computing saturations. We have introduced several optimizations over the original algorithm, for example avoiding the explicit computation of Bass numbers. Still, some free resolutions are computed. In the non-normal case many mixed integer linear programs need to be solved.

References

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