# SignVectors: a SageMath Package for Oriented Matroids

Marcus S. Aichmayr
Institute of Mathematics, University of Kassel
aichmayr@mathematik.uni-kassel.de

## 1 Introduction

We present a SageMath [10] package SignVectors that allows various computations for oriented matroids including cocircuits, topes, face lattices, chirotopes, and dual oriented matroids. Several constructions for oriented matroids are available, for instance using a matrix, a chirotope or the circuits as input. In particular, it can compute realizable oriented matroids over arbitrary computable ordered fields, e.g. the algebraic real numbers. For more details, see also [1]. Our package is used in [2] for analyzing chemical reaction networks using sign vector conditions. The implementation of SignVectors is object-oriented and the code is designed to be maintainable and extensible. The classes take advantage of lazy computation and caching to be efficient. The fully documented package is available on GitHub along with several examples:
`https://github.com/MarcusAichmayr/sign_vectors`

The OrientedMatroids package (version 0.2) [5] for SageMath is still in beta stage with several features not yet implemented. While it offers classes that implement and verify several axiom systems for oriented matroids, our package focuses on efficient computations and constructions and assumes valid input. The C++ package TOPCOM (version 1.1.2) [8] provides tools for triangulations of point configurations and oriented matroids. Similar to our package, it offers commands to compute the cocircuits from a matrix and from a chirotope. However, for realizable oriented matroids, it requires integer input. In Table 1, we provide runtime comparisons for computing the circuits of different realizable oriented matroids represented by dense matrices using an older laptop. We also provide code to reproduce our computational experiments:
`https://github.com/MarcusAichmayr/sign_and_elementary_vector_applications`

Table 1: Chirotope entry counts and runtimes (in seconds) for circuit computation, averaged over 10 matrices over the integers and the real algebraic numbers.

| Ring | Size | Chirotope | ours | TOPCOM | OrientedMatroids |
|---|---|---|---|---|---|
| $\mathbb{Z}$ | $5 \times 10$ | $3 \cdot 10^2$ | 0.013 | 0.017 | 2.4 |
| | $5 \times 20$ | $2 \cdot 10^4$ | 1.6 | 1.2 | 3600 (timeout) |
| | $10 \times 20$ | $2 \cdot 10^5$ | 11 | 42 | – |
| $\overline{\mathbb{Q}} \cap \mathbb{R}$ | $5 \times 10$ | $3 \cdot 10^2$ | 0.95 | – | 2.0 |
| | $5 \times 15$ | $3 \cdot 10^3$ | 14 | – | crash |

# 2 Oriented Matroids

For a short overview on oriented matroids, see [9], and for a comprehensive introduction, we refer to [4, 6]. An oriented matroid is a combinatorial structure that abstracts the properties of, for instance, vector configurations, directed graphs and, in particular, sign vectors of real subspaces. There are various axiom systems for oriented matroids. We explain the *covector axioms* and motivate them using real subspaces.

Let $E$ be a finite set, e.g., $E = \{1, \dots, n\}$. A *sign vector* of a vector in $\mathbb{R}^E$ is obtained by applying the sign function to its components, yielding an element in $\{-, 0, +\}^E$. The set of all sign vectors of a real subspace gives rise to a (realizable) oriented matroid. Clearly, it contains the zero sign vector and, with each element, also its negative. For two real vectors $x$ and $y$, adding a small positive multiple of $y$ to $x$ preserves the signs of the nonzero components of $x$, while the zero components take the corresponding signs of $y$. Therefore, for two sign vectors $X, Y \in \{-, 0, +\}^E$, the oriented matroid also contains their *composition* $X \circ Y$, defined for $e \in E$ by

$$(X \circ Y)_e = \begin{cases} X_e, & \text{if } X_e \neq 0, \\ Y_e, & \text{otherwise.} \end{cases}$$

Next, we consider the line segment between $x$ and $y$. If the vectors have opposite signs in a certain component, there is a point on the line segment such that this component is zero. This property is formalized in the covector elimination axiom below, where we need the set of separating elements $D(X, Y) = \{e \in E : X_e = -Y_e \neq 0\}$. In summary, oriented matroids can be defined by the following four covector axioms:

Let $\mathcal{F} \subseteq \{-, 0, +\}^E$ be a set of sign vectors on the ground set $E$. Then, the pair $\mathcal{M} = (E, \mathcal{F})$ is an *oriented matroid* if $\mathbf{0} \in \mathcal{F}$; for $X \in \mathcal{F}$, $-X \in \mathcal{F}$; for $X, Y \in \mathcal{F}$, $X \circ Y \in \mathcal{F}$; and for $X, Y \in \mathcal{F}$ and $e \in D(X, Y)$, there exists $Z \in \mathcal{F}$ with $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in E \setminus D(X, Y)$.

For a subspace $\mathcal{V} \subseteq \mathbb{R}^E$, the pair $\mathcal{M} = (E, \operatorname{sign} \mathcal{V})$, where $\operatorname{sign} \mathcal{V} = \{\operatorname{sign} v : v \in \mathcal{V}\}$, is an oriented matroid. Such an oriented matroid is called *realizable*.

For sign vectors $X$ and $Y$ on the ground set $E$, we define $X \preceq Y$ if, for all $e \in E$, $X_e \neq 0$ implies $X_e = Y_e$. This partial order induces a lattice structure (the big face lattice) on the set of sign vectors of an oriented matroid (by adding an artificial top element $\mathbf{1}$). The minimal nonzero elements are called *cocircuits* and the maximal sign vectors are called *topes*. The cocircuits (and also the topes) uniquely determine the oriented matroid. We define a matrix $P \in \mathbb{Q}^{2 \times 5}$ and compute the corresponding (realizable) oriented matroid:

```
sage: from sign_vectors import *
sage: P = matrix([[1, 2, 0, 0, 0], [0, 1, 2, -3, 0]])
sage: om = OrientedMatroid(P)
sage: om
Oriented matroid of rank 2 with elements of size 5.
```

We compute the cocircuits of the oriented matroid:

```
sage: om.cocircuits()
{(-0+-0), (0++-0), (--000), (++000), (0--+0), (+0-+0)}
```

To compute the topes from the cocircuits, the package implements [6, Algorithm 1.5].

```
sage: om.topes()
{(---+0), (+--+0), (--+-0), (-++-0), (++-+0), (+++-0)}
```

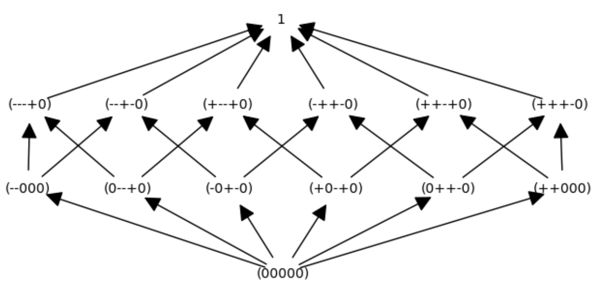
If two real vectors are orthogonal, either they have disjoint support or there exist components whose products have opposite signs. This observation essentially defines orthogonality for sign vectors and leads to the notion of the *dual* oriented matroid. For an oriented matroid $\mathcal{M} = (E, \mathcal{F})$, the pair $\mathcal{M}^\perp = (E, \mathcal{F}^\perp)$ is an oriented matroid where $\mathcal{F}^\perp$ is the set of sign vectors that are orthogonal to all sign vectors in $\mathcal{F}$. The cocircuits of $\mathcal{M}^\perp$ are the *circuits* of $\mathcal{M}$.

```
sage: om.circuits()
{(0000+), (0000-), (00++0), (-+0+0), (00--0), (+-0-0), (-+-00), (+-+00)}
sage: om.dual()
Oriented matroid of rank 3 with elements of size 5.
```

## 3 Face Enumeration and Big Face Lattice

Every face (covector) of an oriented matroid has a well-defined rank. The face enumeration algorithm from [7] (see also [6, Pseudo-Code 1.3]) considers all faces of the same rank and computes the faces of the next lower rank. Our package not only implements this algorithm but also efficiently determines the big face lattice; see [1].

```
sage: om.faces()
[{(00000)},
 {(-0+-0), (0++-0), (--000), (++000), (0--+0), (+0-+0)},
 {(---+0), (+--+0), (--+-0), (-++-0), (++-+0), (+++-0)}]
sage: om.plot()
```




## 4 Chirotopes and Maximal Minors

Support-minimal elements in the row space of a full-rank matrix can be determined by its maximal minors, i.e., determinants of maximal square submatrices (see for example [1, 3]). Analogously, we can determine the cocircuits from the signs of the maximal minors, that is, the chirotope of the oriented matroid (see [9, Theorem 6.2.3]). Therefore, every oriented matroid can be uniquely described by the chirotope.

Similar to the maximal minors of a full-rank matrix, the chirotope is nonzero, alternating, and satisfies a combinatorial version of the Grassmann-Plücker identity (see [9]). The maximal minors of our matrix $P$ from above are:

```
sage: P.minors(2)
[1, 2, -3, 0, 4, -6, 0, 0, 0, 0]
```

Therefore, the chirotope of our oriented matroid is:

```
sage: om.chirotope()
[+, +, -, 0, +, -, 0, 0, 0, 0]
```

The package also offers tools to create an oriented matroid directly from a chirotope:

```
sage: OrientedMatroid.from_chirotope("00--0+++++", 2, 5)
Oriented matroid of rank 2 with elements of size 5.
```

As the cocircuits determine the oriented matroid, we can also recover the chirotope from them. The literature provides a constructive approach to compute the chirotope from circuits or cocircuits (see [4, Corollary 3.5.12] and the surrounding discussion). Corresponding algorithms are discussed in [1] and our package implements them. We consider an oriented matroid given by circuits. Then, the oriented matroid and its chirotope can be computed:

```
sage: om = OrientedMatroid.from_circuits({"0+-0", "+00-", "0-+0", "-00+"})
sage: om
Oriented matroid of rank 2 with elements of size 4.
sage: om.chirotope()
[+, +, 0, 0, -, -]
```

# References

[1] Marcus S. Aichmayr. *Computing Circuits and Sign Vectors with Applications to Linear Inequality Systems and Reaction Networks.* PhD thesis, University of Kassel, Germany, 2026. Submitted.

[2] Marcus S. Aichmayr, Stefan Müller, and Georg Regensburger. A SAGEMATH package for elementary and sign vectors with applications to chemical reaction networks. In *Mathematical software—ICMS 2024*, volume 14749 of *LNCS*, pages 155–164. Springer, Cham, 2024.

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[4] Anders Björner, Michel Las Vergnas, Bernd Sturmfels, Neil White, and Günter Ziegler. *Oriented matroids.* Cambridge: Cambridge University Press, 2nd ed. edition, 1999.

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[9] Jürgen Richter-Gebert and Günter M. Ziegler. Oriented matroids. In *Handbook of discrete and computational geometry*, pages 111–132. Boca Raton, FL: CRC Press, 2017.

[10] SageMath. Sagemath, 2026. https://sagemath.org.